MMP Learning Seminar.

Week 54:

Shokurov's boundary property.

K-trivial fibrations: $f : (X, B) \longrightarrow Y$, where f is surjective between proper normal varieties where (X, B) is a log pair.:

(1) (X, B) is klt over the generic point of Y.

(2) $\operatorname{rank} f_* \mathcal{O}_X(\lceil A(X,B) \rceil) = 1$.

$\varphi^*(K_X + B) = K_Y + B_Y$

(3) There exists a positive integer r, a rational function $\varphi \in K(X)^\times$ and a $\mathbb{Q}$-Carbier divisor D on Y such that:

$$K_X + B + \frac{1}{r}(\varphi) = f^* D.$$

Discrepancy b-divisor: Let (X, B) be a log pair $Y \xrightarrow{\varphi} X$ a projective birational morphism, we can define:

$$A(X,B)_Y := K_Y - \varphi^*(K_X + B).$$

Then $A(X,B)$ is a b-divisor, defined by the formula:

$$A(X,B) = K - \overline{(K_X + B)}.$$

the canonical of the model Y

pull-back of $K_X + B$ to the model Y

Log discrepancy: $E \subseteq Y$ prime divisor.

$$\alpha(E; X, B) := 1 + \mathrm{mult}_E(A(X,B)).$$

trace of the b-divisor on Y.

The minimal log discrepancy of (X,B) in a proper closed subset $W \subseteq X$ is

$$\alpha(W; X, B) := \inf_{c_X(E) \subseteq W} \alpha(E; X, B).$$

$$a(S; X, B) := 1 - \mathrm{mult}_S B.$$

$$a(x; X, B) = \dim \bar{x}.$$

Theorem 1: Let $f : (X,B) \to Y$ klt fiber space & $\dim Y = 1$. Then the moduli $\mathbb{Q}$-divisor M_Y is semiample.

Theorem 2: Let $f : (X,B) \to Y$ be a klt fiber space. Then there exists a proper birational morphism $Y' \longrightarrow Y$ with the following conditions:

(1) $K_{Y'} + B_{Y'}$ is $\mathbb{Q}$-Cartier & $\nu^*(K_{Y'}+B_{Y'}) = K_{Y''} + B_{Y''}$ for every proper birational morphism $\nu : Y'' \longrightarrow Y'$. $\longleftarrow$ descends on Y'

(2) $M_{Y'}$ is a nef $\mathbb{Q}$-divisor & $\underline{\nu^*(M_{Y'}) = M_{Y''}}$.

Remark: The b-divisors $K+B$ & M are Cartier b-divisor

Lemma: $X \dashrightarrow X'$ birational map over Y & $(X,B) \longrightarrow Y$ is a klt fiber space. Then there exists a $\mathbb{Q}$-divisor B' on X' s.t. (X,B) & (X',B') are log crepant and they induce the same discriminant divisor on Y.

$$K_X + B = f^*(K_Y + \underline{B_Y} + \underline{M_Y})$$

$$K_{X'} + B' = f'^*(K_Y + \underline{B_Y} + \underline{M_Y}).$$

Theorem (Inversion of adjunction): Let $f:(X,B)\to Y$ be a K-trivial fibration. There exists N a positive integer:

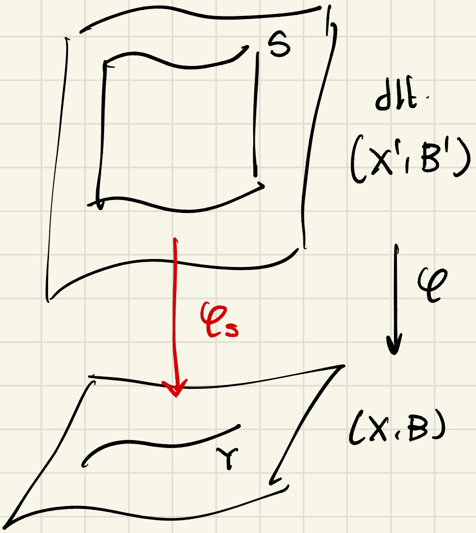
$$\frac{1}{N}\, a(f^{-1}(Z);X,B) \leq a(Z;Y,B_Y) \leq a(f^{-1}(Z);X,B)$$

for every closed subset $Z\subseteq Y$.

In particular, (Y,B_Y) is klt (lc) in a neighborhood of a point y.

$\Longleftrightarrow$ (X,B) is klt (lc) in a neighborhood of $f^{-1}(y)$.

Remark: (X,B) lc pair $Y\subseteq X$ normal lcc.



$S\subseteq \lfloor B'\rfloor$ s.t S maps onto Y.

$K_{X'}+B' = \varphi^*(K_X+B)$

$K_{X'}+B' \sim_{\mathbb{Q},X} 0$.

$(K_{X'}+B')|_S = K_S+B_S \sim_{\mathbb{Q},Y} 0$

(S,B_S) is a K-trivial fibration

$$\begin{tikzcd} (S,B_S) \arrow[d] \\ Y \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, "\varphi_S"'] & X' \arrow[d, "\varphi"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

$K_S+B_S = \varphi_S^*(K_Y+B_Y+M_Y)$

$\Longrightarrow$ $K_X+B|_Y \sim_{\mathbb{Q}} K_Y+B_Y+M_Y$.

Covering tricks & base change:

Theorem 4.1: X smooth q.p. $D \subseteq X$ snc divisor. N a positive integer. There exists a finite Galois cover $\tau : \tilde{X} \longrightarrow X$ satisfying:

(1) $\tilde{X}$ smooth q.p. Σ'_X snc. s.t τ étale outside Σ'_X and $\tau^{-1}(\Sigma'_X)$ snc

(2) the ramification index at every prime component of D is divisible by N.

Sketch: $H_1^{(i)}, \ldots, H_n^{(i)} \in |NA - D_i|$, D_i is a comp of D.

A very ample divisor $\Sigma'_X := D + \sum_{i,j} H_j^{(i)}$ is snc on X.

$X = \bigcup U_\alpha$, $D_i + H_j^{(i)} = (\varphi_{j\alpha}^{(i)})$ on U_α

$L = K(X)[(\varphi_{j\alpha}^{(i)})^{\frac{1}{N}} : i,j]$, $\tilde{X}$ is the normalization of X in the field L. $\square$

Remark 4.2: $\rho : Y \longrightarrow X$ surjective morphism.

from Y smooth $\rho^{-1}(D)$ snc.

Then $\tau : \tilde{X} \longrightarrow X$ a finite cover as in 4.1 . s.t.:

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\tau"'] & \tilde{Y} \arrow[l, "g"'] \arrow[d, "\nu"] \\
X & Y \arrow[l, "\rho"]
\end{tikzcd}$$

i) ν is finite and g projective

ii) $\tilde{Y}$ is smooth q.p

iii) Σ_Y snc on Y, s.t ν is étale over $Y \setminus \Sigma_Y$, $\nu^{-1}(\Sigma_Y)$ has snc

$$\rho^{-1}(\Sigma_X) \subseteq \Sigma_Y.$$

Idea: $H_j^{(i)}$ general enough such that $\rho^{-1}(D + \Sigma_{ij} H_j^{(i)})$ snc

Thm 4.3 (semistable reduction in codim 1):

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times_Y Y' \arrow[l] \arrow[d] & X' \arrow[l] \arrow[dl, "f'"] \\
Y & Y' \arrow[l, "\pi"]
\end{tikzcd}$$

surj morphism of smooth varieties

finite surj.

f' is semistable over codimension one points.

Hodge theoretic input on the cbf:

Theorem 4.4: $f: X \longrightarrow Y$ projective morphism of smooth varieties semistable in codimension one Σ^f_Y snc so that f smooth over $Y \setminus \Sigma^f_Y$. Then:

(1) $f_* \omega_{X/Y}$ is locally free on Y.

(2) $f_* \omega_{X/Y}$ is semipositive

(3) $f_* \omega_{X/Y}$ commutes with base change

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] & Y' \arrow[l, "\rho"'] \end{tikzcd}$$

$$\rho^* f_* \omega_{X/Y} \simeq f'_* \omega_{X'/Y'}.$$

Idea of the proof: $H_0 := R^d f_* \mathbb{Q}_{X_0} \otimes_{\mathbb{Q}_{Y_0}} \mathcal{O}_{Y_0}$ locally free and supports a VHS of weight d on Y_0. $F^d H_0 = f_* \omega_{X_0/Y_0}$.

f semistable in codim $\Longrightarrow$ H_0 has unipotent monodromy around comp of Σ^f_Y.

H canonical extension of H_0. The injection

$$f_* \omega_{X/Y} \hookrightarrow j_*(F^d H_0) \cap H.$$ is an isom

so $f_* \omega_{X/Y}$ is a locally free sheaf. (1).

(2) Griffith's semipositivity theorem of $j_*(F^d H_0)$.

(3) canonical extension commutes with base change. $\square$

Theorem 4.5: Let $f: X \longrightarrow Y$ be a contraction from a smooth projective variety X to a projective curve Y.

E a locally free quotient of $f_* \omega_{X/Y}$.

If $\deg(\det(E)) = 0$, then $\det(E)^{\otimes m} \simeq \mathcal{O}_Y$ for some m.

Shokurov's boundary property:

Lemma 5.1: The discriminant/boundary part commutes with finite base change of Y.

$$\begin{tikzcd}
(X,B) \arrow[d, "f"'] & \tilde{X} \arrow[l, "\pi"'] \arrow[dl, "\tilde{f}"'] & (V,B_V) \arrow[l] \arrow[dll, "h"] \\
Y & &
\end{tikzcd}$$

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

$\tilde{X}$ norm of X

in $K(X)(\varphi^{\frac{1}{b}})$.

$\to$ snc divisor

$(V, B_V) \dashrightarrow (X,B)$ crepant. , $\Sigma_X \subseteq X$, $\Sigma_V \subseteq V$, $\Sigma_Y \subseteq Y$

f, h are smooth over $Y \setminus \Sigma_Y$. Σ_X^h, Σ_V^h are smooth over $Y \setminus \Sigma_Y$.

B, B_V, B_Y, M_Y supported $\Sigma_X, \Sigma_V, \Sigma_Y$

$(1) + (2) \Longleftrightarrow \lceil -B_F \rceil$ effective and $\dim_K H^0(F, \lceil -B_F \rceil) = 1$

Lemma 5.2: In the previous context

$$\begin{tikzcd} (X,B) \arrow[d, "f"'] & V \arrow[l] \arrow[dl, "h"] \\ Y & \end{tikzcd}$$

The following conditions are satisfied:

1) $K(V)/K(X)$ is Galois with Galois group G cyclic of order b.
there exists $\psi \in K(V)^\times$ s.t $\psi^b = \varphi$ and the generator of G
acts by $\psi \longmapsto \zeta\psi$, $\zeta \in K$ is a fixed primitive b-th root of unity.

2) $h: (V, B_V) \longrightarrow Y$ K-trivial except for the rank condition

3) f & h induce the same discriminant divisor.

4) eigensheaf of $h_* \mathcal{O}_V(K_{V/Y})$ corresponding to $)$ is

$$\mathcal{L} := f_* \mathcal{O}_X(\lceil -B + f^*B_Y + f^*M_Y \rceil) \cdot \psi.$$

5) If $h: V \longrightarrow Y$ is semistable in codim 1,
then M_Y is integral, $\mathcal{L}$ is semipositive, and $\mathcal{L} = \mathcal{O}_Y(M_Y) \cdot \psi$.

$$\begin{tikzcd}
 & V \arrow[dl, "g"'] \arrow[dd] & & V' \arrow[ll] \arrow[dl, "g'"'] \arrow[ddl] \\
X \arrow[dr, "f"'] & & X' \arrow[ll] \arrow[d, "f'"'] & \\
 & Y & Y' \arrow[l] &
\end{tikzcd}$$

Fact: the diagram

$$\begin{tikzcd}
V' \arrow[r] \arrow[rr, bend left] & X' \arrow[r] & Y'
\end{tikzcd}$$

also satisfies condition (i) - (v)

$Y \leftarrow Y'$ is a finite base change étale outside Σ_Y.

Proposition 5.4: There exists $Y' \longrightarrow Y$ Galois base change s.t $V' \longrightarrow Y'$ is semistable in codimension

Proposition 5.5: $Y' \xrightarrow{\gamma} Y$ generically finite proj morphism from Y' smooth. Assume there exists $\Sigma_{Y'}$ snc on Y' which contains $\gamma^{-1}(\Sigma_Y)$ and the locus where γ is not étale. Let $M_{Y'}$ be the moduli part of $(V', B_{V'}) \dashrightarrow (X', B') \longrightarrow Y'$. Then $\gamma^*(M_Y) = M_{Y'}$.

$$\begin{tikzcd}
 & V \arrow[ld, "g"'] \arrow[dd] & & V' \arrow[ll] \arrow[ld, "g'"'] \arrow[ldd] \\
X \arrow[rd, "f"'] & & X' \arrow[ll] \arrow[d, "f'"'] & \\
 & Y & Y' \arrow[l] &
\end{tikzcd}$$

Proof: Step 1: V/Y & V'/Y' are semistable in codimension one.

Thm 4.4: $h'_*\mathcal{O}_{V'}(K_{V'/Y'}) \simeq \gamma^*(h_*\mathcal{O}_V(K_{V/Y}))$

5.2 implies $\gamma^*\mathcal{O}_Y(M_Y) = \mathcal{O}_{Y'}(M_{Y'})$

$\gamma^* M_Y - M_{Y'}$ is ~ 0 and exceptional over Y. Then $\gamma^* M_Y = M_{Y'}$.

Step 2:

$$\begin{tikzcd}
\bar{Y} \arrow[d, "\tau"'] & \bar{Y}' \arrow[l, "\gamma'"'] \arrow[d, "\tau'"] \\
Y & Y' \arrow[l, "\gamma"]
\end{tikzcd}$$

$\left.\begin{array}{l}\bar{V}/\bar{Y} \\ \bar{V}'/\bar{Y}'\end{array}\right\}$ semistable in cod 1.

$M_{\bar{Y}'} = \gamma'^*(M_{\bar{Y}})$ by step 1.

τ & τ' are finite morphisms, so 5.1 $\Longrightarrow$ $\tau^*(M_Y) = M_{\bar{Y}}$
& $\tau'^*(M_{Y'}) = M_{\bar{Y}'}$.

Therefore $\tau'^*(M_{Y'} - \gamma^*(M_Y)) = 0$ this implies

$$M_{Y'} = \gamma^* M_Y.$$

$\square$

Proof of Theorem 2:

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y).$$

Assume X smooth, V = resol of norm of X in $K(X)(\varphi^{\frac{1}{b}})$.

Σ_f on Y s.t $(V, B_V) \longrightarrow (X, B) \longrightarrow Y$ satisfies (i)-(v)

except at Σ_f.

$Y' \longrightarrow Y$, $\Sigma_{Y'} := \sigma^{-1}(\Sigma_f)$ is a snc divisor

We have an induced set-up $(V', B_{V'}) \dashrightarrow (X', B') \longrightarrow Y'$.

Claim: $\sigma^*(M_{Y'}) = M_{Y''}$ for every $Y'' \xrightarrow{\sigma} Y'$.

Hence $\sigma^*(K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''}$.

$$\begin{tikzcd}
Y'' \arrow[d, "\sigma"'] & Y''' \arrow[l] \arrow[dl, "\sigma'"] \\
Y' &
\end{tikzcd}$$

By Proposition 5.5,

$\sigma'^*(M_{Y'}) = M_{Y'''}$ $\Downarrow$

$\sigma^*(M_{Y'}) = M_{Y''}$. $\square$.

$T: \bar{Y}' \longrightarrow Y'$ be a covering given by 5.4 & 5.2.

$M_{\bar{Y}'}$ is Cartier and $\mathcal{O}_{\bar{Y}'}(M_{\bar{Y}'})$ is an invertible semipositive shf.

$\longrightarrow$ $M_{\bar{Y}'}$ is nef. $T^*(M_{Y'}) = M_{\bar{Y}'}$ according to Lemma 5.1. $\Longrightarrow$ $M_{Y'}$ is nef. $\square$

$$\begin{tikzcd}
(X, B) \arrow[d] & \text{log pair} \arrow[dd, squiggly] & (X, \overset{\text{eff}}{B} + \overset{\text{push-forward of nef}}{M}) \\
Y & & \\
K_Y + B_Y + M_Y & \text{generalized log pair} \arrow[l] &
\end{tikzcd}$$

$$K_Y + \underset{\substack{\downarrow \\ \text{boundary divisor}}}{B_Y} + \underset{\substack{\downarrow \\ \text{nef on a higher model}}}{M_Y}$$

$$\begin{tikzcd}
(X, B + M) \arrow[d] \\
(Y, B_Y + M_Y)
\end{tikzcd}$$

inversion

$$K_X + B + M = \varphi^*(K_Y + B_Y + M_Y)$$